

Field Theory in Extra Dimensions*

D.I.Kazakov

*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research,
Dubna, Russia*

and

Institute for Theoretical and Experimental Physics, Moscow, Russia

Abstract

We analyze the possibility to construct a self-consistent gauge field theory in $D > 4$. We first look for the cancellation of the UV divergences in SUSY theories. Then, following the Wilson RG approach, we study the RG equation for the gauge coupling in perturbative and nonperturbative regimes. In the first case the power law running is discussed. In the second case it is shown that there exist the ultraviolet fixed point where the gauge coupling is dimensionless in any space-time dimension. This fixed point is nonperturbative and corresponds to scale invariant theory. The same phenomenon also happens in supersymmetric theory in $D=6$.

1 Introduction

The extra dimensional theories have already become popular for a few years [1, 2]. The main motivation for introducing extra space dimensions comes from the string theory which prefers to live in 26, 10 or 11 dimensions [3]. However, the string theory is still far from being completed and the link to the lower energy theory described by usual QFT is still missing. Staying in the framework of QFT one may wonder whether this extra dimensional theory can be consistent in any sense. Since by general power counting any interacting field theory (except for ϕ^3 in $D = 6$) is nonrenormalizable, it looks hardly possible.

At the same time, following the concept of effective theory one requires a consistency only up to a given order of perturbative expansion, thus accepting the nonrenormalizable theories. Still, the renormalizability or the possibility to consider, in principle, a self-consistent theory served as a driving force in construction of the Standard Model and should not be underestimated. Whether or not this attitude may be applied to the extra dimensional theories remains an open question.

In this talk I would like to concentrate on the following questions:

- Can one construct a self-consistent QFT in $D > 4$?

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- Is it possible to get rid of UV divergences ?
- Power law running: is it reliable ?
- Nonperturbative fixed points: do they exist?

2 UV divergences in SUSY theories for arbitrary D .

In principle, there is a chance that all the UV divergences cancel each other, like it takes place in $N = 4$, 2 and even $N = 1$ SUSY field theories in $D = 4$ [4], and one might have a consistent theory. This possibility has been studied in the literature [5]-[11].

Indeed, since one has an equivalence between extended supersymmetry in $D=4$ and reduced supersymmetry in higher dimensions, namely

$$N = 4, D = 4 \Leftrightarrow N = 2, D = 6 \Leftrightarrow N = 1, D = 10,$$

$$N = 2, D = 4 \Leftrightarrow N = 1, D = 6,$$

some cancellations take place. For example, total cancellation of the quadratic and logarithmic UV divergences in $N = 4$ $D = 4$ SUSY theory leads to the cancellation of the quartic and quadratic divergences in $N = 2$ $D = 6$ theory and octic and sextic divergences in $N = 1$ $D = 10$ case. Analogously the condition which guarantees the cancellation of the logarithmic divergences in $N = 2$ $D = 4$ theory works also for the quadratic divergences in $N = 1$ $D = 6$ case. The results are summarized below (Here C_A and T_R are the Casimir operators of the adjoint and arbitrary matter field representations, respectively.)

D	N	UV Divergences in one loop order	
$D = 4$	$N = 1$	$-11/3 C_A + 2/3 C_A + 2/3 T_R + 1/3 T_R$	$= -(3C_A - T_R)$
$\log \Lambda^2$	$N = 2$	$-11/3 C_A + 2/3 C_A + C_A + 2 T_R$	$= -2(C_A - T_R)$
	$N = 4$	$-2 C_A + 2 C_A$	$= 0$
$D = 6$	$N = 1$	$-10/3 C_A + 4/3 C_A + 4/3 T_R + 2/3 T_R$	$= -2(C_A - T_R)$
Λ^2	$N = 2$	$-2 C_A + 2 C_A$	$= 0$
$D = 10$	$N = 1$	$-8/3 C_A + 8/3 C_A$	$= 0$
Λ^6			

Table 1: One loop UV divergences in SUSY gauge theories for arbitrary D

One can see that the leading divergences indeed cancel each other. However, this is not true anymore for the logarithmic divergences. Strictly speaking they are not gauge invariant and to get the gauge invariant statement one has to go on shell. Then at lower orders the divergences indeed cancel [8, 9, 11], but in higher orders they may well appear being unprotected by any symmetry [10]. Indeed, it has been checked by explicit calculation in components [8, 9] that $D = 6$ $N = 1$ SUSY gauge theory is on-shell finite up to two loops. However, within the (constrained) superfield formalism it is possible to show that the nonvanishing invariants in higher loops exist [10].

Thus, the theory remains *perturbatively nonrenormalizable*.

3 Wilsonian RG in $D > 4$.

We now look for the alternative possibilities to construct a viable higher dimensional theory. We follow the so-called Wilson Renormalization Group approach [12].

Consider first the usual gauge theory in D dimensions

$$\mathcal{L} = -\frac{1}{4}Tr F_{\mu\nu}^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]. \quad (1)$$

The fields and the coupling have the following canonical dimensions:

$$[A] = \frac{D-2}{2}, \quad [F] = \frac{D}{2}, \quad [g] = 2 - \frac{D}{2}.$$

Thus, $D = 4$ is the critical dimension: the coupling is dimensionless, the operators are marginal and the theory is renormalizable in a usual sense.

To go beyond the critical dimension we follow the standard approach [13] (see also Ref.[14]) based on dimensional regularization and analytical continuation. Consider the dimensionless quantity

$$\tilde{g} \equiv g\mu^{D/2-2} \quad \Rightarrow \quad [\tilde{g}] = 0,$$

where μ is some scale, and expresses the bare coupling in terms of a renormalized one

$$g_B = \mu^{2-D/2} \tilde{g} Z_g(\tilde{g}), \quad (2)$$

where the renormalization constant Z_g depends on D and in the minimal subtraction scheme contains only the pole terms in $(D - D_c)$ with $D_c = 4$ in this case.

The crucial point here is that *the renormalization constant Z_g depends on \tilde{g} and does not contain the infinite number of higher dimensional operators that may appear at $D > D_c$* . In Wilson approach these operators are irrelevant while going toward the *infrared* direction and may be ignored [12].

Differentiating then eq.(2) with respect to a scale keeping g_B fixed one gets

$$0 = (2 - \frac{D}{2})\tilde{g} + \beta(\tilde{g}) - \tilde{g}\gamma_g(\tilde{g}), \quad (3)$$

where as usual $\beta(\tilde{g}) = \mu \frac{d}{d\mu} \tilde{g}|_{g_B}$ and $\gamma_g = -\mu \frac{d}{d\mu} \ln Z_g|_{g_B}$. This leads to the following RG equation for the coupling

$$\mu \frac{d}{d\mu} \tilde{g} = \beta(\tilde{g}) = \tilde{g}(\frac{D}{2} - 2 + \gamma_g). \quad (4)$$

In general $\gamma_g(g)$, and hence $\beta(g)$, may depend on D being finite while $D \rightarrow D_c$. However, in the MS-scheme this dependence is absent and γ_g can be calculated directly in the critical dimension. We use this advantage since while the β function and the anomalous dimensions are scheme dependent, the value of the anomalous dimension at the critical point is universal. This means that it can be calculated in any scheme. It is useful to proceed in the background field gauge where $\gamma_g = \frac{1}{2}\gamma_A$ the latter being the gauge field anomalous dimension.

4 A perturbative solution

Consider now the perturbative solution to eq.(4). For this purpose we take the one loop expression for $\gamma_A = b\tilde{g}^2$. Then the solution looks like

$$\frac{1}{\tilde{g}^2} = \frac{1}{\tilde{g}_0^2} \left(\frac{\mu^2}{Q^2} \right)^\varepsilon + \frac{b}{2\varepsilon} \left[\left(\frac{\mu^2}{Q^2} \right)^\varepsilon - 1 \right], \quad \varepsilon = \frac{D}{2} - 2, \quad (5)$$

and exhibits the power law running. When $D \rightarrow 4$ or $\varepsilon \rightarrow 0$ one comes back to the usual logarithmic behaviour

$$\frac{1}{\tilde{g}^2} = \frac{1}{\tilde{g}_0^2} + \frac{b}{2} \log \frac{\mu^2}{Q^2}. \quad (6)$$

Eq.(5), being obtained though on the basis of Kaluza-Klein approach, has been considered as the way to the low scale unification in Grand Unified Theories [15]. Indeed, if one assumes that at some energy scale the extra dimension comes into play, one has to switch at this scale from the log running in $D = 4$ into the power running in $D > 4$ and gets the lower scale unification as shown in Fig.1.

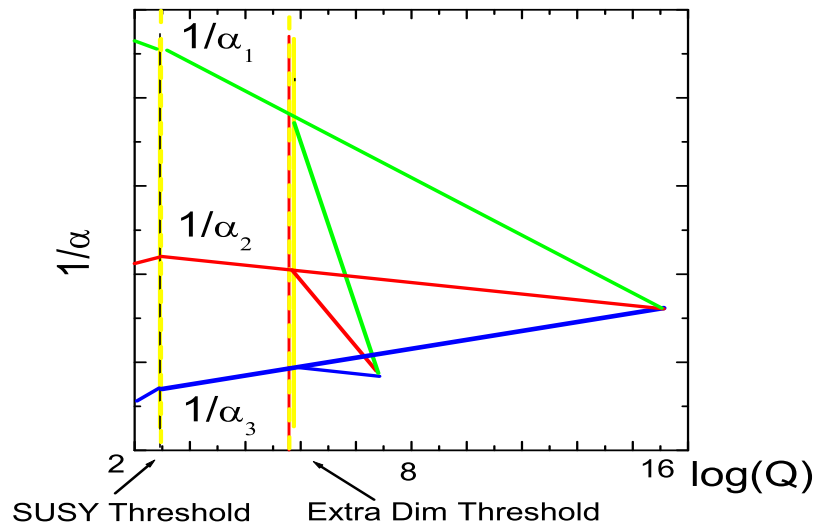


Figure 1: Power law versus logarithmic unification of the couplings

Note that the very fact of unification of the three curves according to eq.(5) does not depend on ε and if the curves unify for $D = 4$ they will do it for any D .

This very appealing picture, however, suffers one substantial problem. Strictly speaking this power law running is valid if going the *infrared* direction. Just in this case one can ignore the infinite number of higher dimensional operators which are irrelevant here. When going ultraviolet direction they all become relevant and eq.(4) with a single coupling is no more valid. This irreversibility of the RG equations in the Wilson approach is essential and can not be ignored. One may hope to overcome this difficulty in some underlying theory, like the string one, however, the result is unclear. Therefore, to my mind, the advocated power law running of the couplings with the low scale unification in the UV region *can not* be considered as reliable so far.

5 Nonperturbative fixed point

Consider now the nonperturbative solution to eq.(4). It has two fixed points

- 1) $\tilde{g} = 0 \rightarrow g = 0, \quad \gamma_A = 0,$
- 2) $\tilde{g} = g^*, \quad \gamma_A = 4 - D.$

The first one is trivial, this is the so-called Gaussian fixed point, it is perturbative. The second one is nonperturbative, it is the so-called Wilson-Fisher fixed point [12]. The anomalous dimension here is not small, it is integer. It is achieved at the value of the coupling which is unknown, though the value of the anomalous dimension is known *exactly*. Since the anomalous dimension in gauge theories, contrary to the scalar case, is negative, the fixed point of the second kind exists for $D > 4$. (see Fig.2).

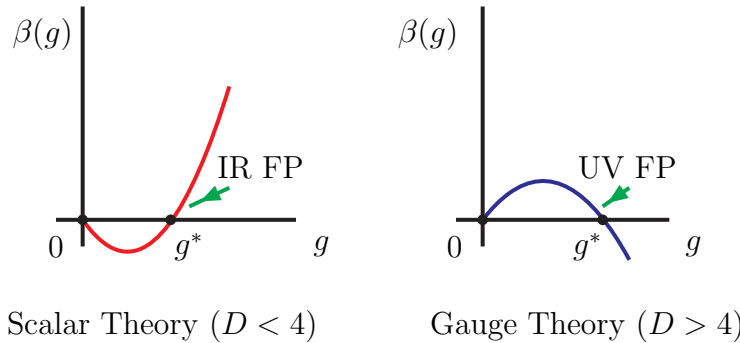


Figure 2: The fixed points in scalar and gauge theories

Such a fixed point in a gauge theory within the ϵ -expansion has been advocated in ref.[16]. Some additional supporting arguments in favour of the fixed point in 5 dimensions come from the lattice calculations [17]. At last, there is also very useful analogy between the gauge theory and the two dimensional nonlinear sigma-model. The latter has a fixed point in the leading order [18] which is also true within the $1/N$ expansion performed directly in three dimensions [19].

Consider the properties of the fixed point #2. Let us calculate the dimensions. One has for the field

$$[A] = \frac{D-2}{2} + \frac{1}{2}\gamma_A = \frac{D-2}{2} + \frac{4-D}{2} = 1$$

in any D . To calculate the dimension of the coupling, one has to consider the vertex $g\partial A[A, A]$ which gives

$$D = [g] + 1 + 3[A] + \gamma_V.$$

Since $\gamma_V = -\gamma_A$ in the background gauge, one obtains

$$[g] = D - 4 - \gamma_V = D - 4 + \gamma_A = 0 \quad \text{in any } D \quad !$$

Thus, one has a dimensionless coupling at the fixed point that means renormalizability. *The theory at the fixed point is perturbatively nonrenormalizable, but nonperturbatively renormalizable!* [20, 21]). The existence of a renormalizable field theory beyond

PT relies, in the sense of statistical physics, on the existence of a fixed point (see e.g Ref. [14], p.549).

Since the full dimension of the field is known, it is possible to calculate the behaviour of the propagator at the fixed point. One has

$$\widehat{AA} \sim \frac{1}{(x^2)^1} \Rightarrow \int \frac{d^D x e^{ipx}}{(x^2)} \sim \frac{1}{(p^2)^{\frac{D-2}{2}}} \quad (7)$$

Thus, for instance, for $D = 6$ at the non-Gaussian fixed point the propagator behaves like $1/p^4$, i.e. much faster than in the usual case.

One can try to construct an effective Lagrangian that takes into account the anomalous dimensions calculated above. In $D = 6$, as it is suggested by the one-loop calculation and the behaviour of the propagator, it is

$$\mathcal{L}_{eff} \sim \text{Tr}(D_\mu F_{\mu\nu})^2. \quad (8)$$

The effective Lagrangian (8) has a remarkable property: it is *scale* invariant. Earlier I assumed that it might also be *conformal* invariant [22], though conformal invariance does not necessarily follows from the scale one [23] and has to be checked. If it is true, then this will essentially constrain the Green functions allowing the non-perturbative information of their properties [24].

6 Nonperturbative fixed point in SUSY theories

A similar phenomenon takes place in SUSY gauge theories. Here, however, we are faced with the problem: supersymmetry does not exist in any dimension. Indeed, from the requirement of equal number of the fermionic and bosonic degrees of freedom for the gauge field and its superpartner one gets

$$D - 2 = 2^{[D/2]-1(2)} \quad (9)$$

with the solution $D = 4, 6, 10$ and possible modification for the odd values of D . Moreover, as was already mentioned, higher dimensional supersymmetry is equivalent to extended one in lower dimensions.

Therefore, when considering supersymmetric theory in one of the possible higher dimensions and going to the critical dimension following the minimal subtraction procedure, one has to keep track of degrees of freedom. For instance, choosing $N=1$ $D=6$ theory and going to $D=4$ one has to take $N=2$ SUSY in $D=4$, choosing $N=1$ $D=10$ theory one has to take $N=4$ $D=4$ SUSY, etc.

The number of possibilities, hence, is very limited. One has an extended SUSY theory in $D=4$ with the single coupling: all the Yukawa couplings are equal to the gauge one due to extended supersymmetry.

Remind that in 4 dimensions $N=2$ SUSY theory (in supefield formalism) has only one loop UV divergences, while $N=4$ SUSY theory is totally finite (see Table 1). This means that eq.(4) in SUSY case is essentially simplified. One has

$$\gamma_A = b\tilde{g}^2 \text{ for } D = 6 \text{ and } \gamma_A = 0 \text{ for } D = 10. \quad (10)$$

Consequently, the nonperturbative fixed point defined by the condition $\gamma_A = 4 - D$ exists in $D = 6$ when

$$\tilde{g}^2 = g^{*2} = -\frac{2}{b}, \quad b < 0,$$

and does not exist in $D = 10$ due to vanishing of γ_A in this case. Taking the value of b from the Table 1 one finds

$$b \sim -2(C_A - T_R) < 0 \quad \Rightarrow \quad T_R < C_A.$$

Thus, if the number of hypermultiplets is not big, the fixed point exists. In particular it exists in pure gauge case when $T_R=0$. Remarkably, that in SUSY case one knows not only the anomalous dimension, but the critical coupling as well.

At this fixed point a theory possess all the properties mentioned above. It is perturbatively nonrenormalizable, but nonperturbatively renormalizable and scale invariant. The effective action should be the SUSY generalization of that of eq.(8).

There is another subtlety with supersymmetry. While supersymmetric gauge theory exists for $D \leq 10$, it is known that superconformal algebra is only possible for $D \leq 6$ [25]. Earlier [22] I claimed one can get the fixed point in SUSY gauge theory in any D . However, the present analysis shows the existence of the nontrivial fixed point in $D = 6$, and may be in $D=5$, but not in $D = 10$, that matches the above statement [25] and resolves the contradiction.

These observed fixed points may be related to those originated from the string dynamics for $D=5$ and 6 [26]. We use here the more familiar language that is close to statistical physics and critical phenomena. In a sense we give an explicit example of a local field theory with non-trivial fixed point thus strengthening the claim (based on string theory) that exist field theories that flow to non-trivial fixed points in more than 3 dimensions [26].

7 Conclusion

Summarizing the analysis of the gauge and SUSY field theories in higher dimensions from the point of view of their renormalizability and consistency, we come to the following conclusions

- Perturbative finiteness in $D > 4$ seems not to be valid;
- Within the Wilson RG approach one can write the equation for the couplings in $D > 4$ which exhibits the power law running in the infrared direction, but not in the ultraviolet;
- In this approach there exist the nontrivial nonperturbative fixed point which may lead to nonperturbative renormalizability;
- At the fixed point the theory possesses the scale invariance, and the anomalous dimensions are known exactly;
- The same phenomenon happens in SUSY theories in $D=6$ ($D=5$);

- It is quite possible that at the fixed point $N=1$ $D=6$ SUSY gauge theory, like the $N=4$ $D=4$ one, is also conformal invariant.

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